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ON Minimal λ*-Open Sets

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Abstract. In this paper, we introduce and discuss minimal λ*-open sets in topological spaces. We establish some basic properties of minimal λ*-open sets and provide an example to illustrate that minimal λ*-open sets are independent of minimal open sets. we obtain an application of a theory of minimal λ*-open sets and we defined a λ*-locally finite space.

Index Terms — s-operation, λ^* –open, minimal λ*-open sets, λ-$T_{1/2}$ space , λ*-locally finite space.

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1. Introduction

The study of semi open sets in topological space was initiated by Levine [**9**]. The concept of operation γ was initiated by S. Kasahara [**7**]. He also introduced γ -closed graph of a function. Using this operation, H. Ogata[**8**] introduced the concept of γ -open sets and investigated the related topological properties of the associated topology τ_γ and τ. He further investigated general operator approaches of close graph of mappings. Further S. Hussain and B. Ahmad [**4**] continued studying the properties of γ -open(γ-closed) sets. In 2009, B. Ahmad and S. Hussain [**5**], introduced the concept of minimal gamma γ-open sets. In 2011[**1**] (respt. in 2013[**2**]) Alias B. Khalaf and Sarhad Faiq Namiq defined an operation λ called s-operation and defined and investigated several properties of λ-regular, λ-open, λ-identity, λ-monotone, λ-idempotent and λ-additive operations on the family of semi open sets in topological spaces. They defined λ*-open set [**6**] which is equivalent to λ-open set[**1**] and λ_s- open set[**2**] by using s-operation. They [**6**] defined and investigated several properties of λ*-derived, λ*- interior and λ*-closure points in Topological Spaces. In this paper, we introduce and discuss minimal λ*-open sets in topological spaces.

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We establish some basic properties of minimal λ*-open sets and provide an example to illustrate that minimal λ*-open sets are independent of minimal open sets.

First, we recall some definitions and results used in this paper. We shall write a space in place of a topological space.

2. Preliminaries

Throughout, X denote topological spaces. Let A be a subset of X, then the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$ respectively. A subset A of a topological space (X,τ) is said to be semi open [9] if $A \subseteq Cl(Int(A))$. The complement of a semi open set is said to be semi closed [9]. The family of all semi open (resp. semi closed) sets in a topological space (X,τ) is denoted by $SO(X,\tau)$ or $SO(X)$ (resp. $SC(X,\tau)$ or $SC(X)$). We consider λ as a function defined on $SO(X)$ into $P(X)$ and $\lambda: SO(X) \longrightarrow P(X)$ is called an s-operation if $V \subseteq \lambda(V)$ for each non-empty semi open set V. It is assumed that $\lambda(\phi) = \phi$ and $\lambda(X) = X$ for any s-operation λ.

Definition 2.1.[6] Let (X,τ) be a topological space and $\lambda: SO(X) \longrightarrow P(X)$ be an s-operation, then a subset A of X is called a λ^*-open set which is equivalent to λ –open set[**1**] and λ_s-open set[**2**] if for each $x \in A$ there exists a semi open set U such that $x \in U$ and $\lambda(U) \subseteq A$.
The complement of a λ^*-open setis said to be λ^*-closed. The family of all λ^*-open (resp., λ^*-closed) subsets of a topological space (X,τ) is denoted by $SO_\lambda(X,\tau)$ or $SO_\lambda(X)$ (resp. , $SC_\lambda(X,\tau)$ or $SC_\lambda(X)$).

Proposition 2.2.[1]. For a topological space $(X,\tau), SO_\lambda(X) \subseteq SO(X)$.

The following examples show that the converse of the above proposition may not be true in general.

Example 2.3.[1]. Let $X = \{a,b,c\}$, and $\tau = \{\phi,\{a\},X\}$. We define an s-operation $\lambda: SO(X) \longrightarrow P(X)$ as $\lambda(A) = A$ if $b \in A$ and $\lambda(A) = X$ otherwise. Here, we have $\{a,c\}$ is semi open set but it is not λ^*-open.

Definition 2.4. [1]. Let (X,τ) be a space, an s-operation λ is said to be s-regular if for every semi open sets U and V of $x \in X$, there exists a semi open set W containing x such that $\lambda(W) \subseteq \lambda(U) \cap \lambda(V)$.

Definition 2.5. [6]. Let (X,τ) be a topological space and let A be a subset of X. Then:

(1) The λ^*-closure of A ($\lambda^*Cl(A) = \lambda Cl(A)$) is the intersection of all λ^*-closed sets containing A.

(2) The λ^*-interior of A ($\lambda^*Int(A) = \lambda\, Int(A)$) is the union of all λ^*-open sets of X contained in A.

Proposition 2.6. [6]. For each point $x \in X, x \in \lambda^*Cl(A) = \lambda Cl(A)$ if and only if $V \cap A \neq \phi$ for every $V \in SO_\lambda(X)$ such that $x \in V$.

Proposition 2.7. [3]. Let $\{A_\alpha\}_{\alpha\in I}$ be any collection of λ^*-open sets in a topological space (X,τ), then $\bigcup_{\alpha\in I} A_\alpha$ is a λ^*-open set.

Proposition 2.8. [2]. Let λ be a s-regular s-operation. If A and B are λ^*-open sets in X, then $A \cap B$ is also a λ^*-open set.

3. Minimal λ*-open sets

Definition 3.1. Let X be a space and $A \subseteq X$ a λ*-open set. Then A is called a minimal λ*-open set if ϕ and A are the only λ*-open subsets of A.

The following Example shows that minimal λ*-open sets and minimal open sets are independent of each other.

Example 3.2. Let $X = \{a,b,c\}$, and $\tau = \{\phi,\{a\},\{b\},\{a,b\},\{a,c\},X\}$. We define an s-operation $\lambda: SO(X) \longrightarrow P(X)$ as $\lambda(A) = A$ if $A = \{a,c\}$ and $\lambda(A) = X$ otherwise. The λ*-open sets are $\phi, \{a,c\}$ and X. Here $\{a\}$ is minimal open set which is not minimal λ*-open. Also $\{a,c\}$ is minimal λ*-open set which is not minimal open.

Proposition 3.3. Let (X,τ)be a topological space. Then:

(1) Let A be a minimal λ*-open set and B a λ*-open set. Then $A \cap B = \phi$ or $A \subseteq B$, where λ is λ -regular.

(2) Let B and C be minimal λ*-open sets. Then $B \cap C = \phi$ or $B = C$, where λ is λ -regular.

Proof. (1) Let B be a λ*-open set such that $A \cap B \neq \phi$. Since A is a minimal λ*-open set and $A \cap B \subseteq A$, we have $A \cap B = A$. Therefore $A \subseteq B$.

(2) If $A \cap B \neq \phi$, then we see that $B \subseteq C$ and $C \subseteq B$ by (1). Therefore $B = C$.

Proposition 3.4. Let A be a minimal λ*-open set. If x is an element of A, then $A \subseteq B$ for any λ*-open neighborhood B of x, where λ is λ -regular.

Proof. Let B be a λ*-open neighborhood of x such that $A \not\subset B$. Since where λ is λ –regular operation, then $A \cap B$ is λ*-open set such that $A \cap B \subseteq A$ and $A \cap B \neq \phi$. This contradicts our assumption that A is a minimal λ*-open set.

Proposition 3.5. Let A be a minimal λ*-open set. Then for any element x of A, $A = \bigcap\{B: B$ is λ*-open neighborhood of $x\}$, where λ is λ-regular.

Proof. By Proposition 3.4, and the fact that A is λ*-open neighborhood of x, we have $A \subseteq \bigcap\{B: B$is λ*-open neighborhood of $x\} \subseteq A$. Therefore we have the result.

Proposition 3.6. Let A be a minimal λ*-open set in X and $x \in X$ such that $x \notin A$. Then for any λ*-open neighborhood C of x, $C \cap A = \phi$ or $A \subseteq C$, where λ is λ-regular.

Proof. Since C is a λ*-open set, we have the result by Proposition 3.3.

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Corollary 3.7. Let A be a minimal λ^*-open set in X and $x \in X$ such that $x \notin A$. Define $A = \bigcap\{B : B$ is λ^*-open neighborhood of $x\}$. Then $A_x \cap A = \phi$ or $A \subseteq A_x$, where λ is λ-regular.

Proof. If $A \subseteq B$ for any λ^*-open neighborhood B of x, then $A \subseteq \bigcap\{B : B$ is λ^*-open neighborhood of $x\}$. Therefore $A \subseteq A_x$. Otherwise there exists a λ^*-open neighborhood B of x such that $B \cap A = \phi$. Then we have $A_x \cap A = \phi$.

Corollary 3.8. If A is a nonempty minimal λ^*-open set of X, then for a nonempty subset C of A, $A \subseteq \lambda^*Cl(C)$, where λ is λ-regular.

Proof. Let C be any nonempty subset of A. Let $y \in A$ and B be any λ^*-open neighborhood of y. By Proposition 3.4, we have $A \subseteq B$ and $C = A \cap C \subseteq B \cap C$. Thus we have $B \cap C \neq \phi$ and hence $y \in \lambda^*Cl(C)$. This implies that $A \cap \lambda^*Cl(C)$. This completes the proof.

Proposition 3.9. Let A be a nonempty λ^*-open subset of a space X. If $A \subseteq \lambda^*Cl(C)$, then $\lambda^*Cl(A) = \lambda^*Cl(C)$, for any nonempty subset C of A.

Proof. For any nonempty subset C of A, we have $\lambda^*Cl(C) \subseteq \lambda^*Cl(A)$. On the other hand, by supposition we see $\lambda^*Cl(A) = \lambda^*Cl(\lambda^*Cl(C))= \lambda^*Cl(C)$ implies $\lambda^*Cl(A) \subseteq \lambda^*Cl(C)$. Therefore we have $\lambda^*Cl(A) = \lambda^*Cl(C)$ for any nonempty subset C of A.

Proposition 3.10. Let A be a nonempty λ^*-open subset of a space X. If $\lambda^*Cl(A) = \lambda^*Cl(C)$, for any nonempty subset C of A, then A is a minimal λ^*-open set.

Proof. Suppose that A is not a minimal λ^*-open set. Then there exists a nonempty λ^*-open set B such that $B \subseteq A$ and hence there exists an element $x \in A$ such that $x \notin B$. Then we have $\lambda^*Cl(\{x\}) \subseteq X \setminus B$ implies that $\lambda^*Cl(\{x\}) = \lambda^*Cl(A)$. This contradiction proves the proposition

Combining Corollary 3.8 and Propositions 3.9 and 3.10, we have:

Theorem 3.11. Let A be a nonempty λ^*-open subset of space X. Then the following are equivalent:

(1) A is minimal λ^*-open set, where λ is λ-regular.

(2) For any nonempty subset C of A, $A \subseteq \lambda^*Cl(C)$.

(3) For any nonempty subset C of A, $\lambda^*Cl(A) = \lambda^*Cl(C)$.

4. Finite λ^*-open sets

In this section, we study some properties of minimal λ^*-open sets in finite λ^*-open sets and λ^*-locally finite spaces.

Proposition 4.1. Let (X, τ) be a topological space and $\phi \neq B$ a finite λ^*-open set in X. Then there exists at least one (finite) minimal λ^*-open set A such that $A \subseteq B$.

Proof. Suppose that B is a finite λ^*-open set in X. Then we have the following two possibilities:

(1) B is a minimal λ^*-open set.

(2) B is not a minimal λ^*-open set.

In case (1), if we choose $B = A$, then the proposition is proved. If the case (2) is true, then there exists a nonempty (finite) λ^*-open set B_1 which is properly contained in B. If B_1 is minimal λ^*-open, we take $A = B_1$. If B_1 is not a minimal λ^*-open set, then there exists a nonempty (finite) λ^*-open set B_2 such that $B_2 \subseteq B_1 \subseteq B$. We continue this process and have a sequence of λ^*-open sets $\ldots \subseteq B_m \subseteq \cdots \subseteq B_2 \subseteq B_1 \subseteq B$. Since B is a finite, this process will end in a finite number of steps. That is, for some natural number k, we have a minimal λ^*-open set B_k such that $B_k = A$. This completes the proof.

Definition 4.2. A space X is said to be a λ^*-locally finite space, if for each $x \in X$ there exists a finite λ^*-open set A in X such that $x \in A$.

Corollary 4.3. Let X be a λ^*-locally finite space and B a nonempty λ^*-open set. Then there exists at least one (finite) minimal λ^*-open set A such that $A \subseteq B$, where λ is λ-regular.

Proof. Since B is a nonempty set, there exists an element x of B. Since X is a λ^*-locally finite space, we have a finite λ^*-open set B_x such that $x \in B_x$. Since $B \cap B_x$ is a finite λ^*-open set, we get a minimal λ^*-open set A such that $A \subseteq B \cap B_x \subseteq B$ by Proposition 4.1.

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Proposition 4.4. Let X be a space and for any $\alpha \in I$, B_α a λ*-open set and $\phi \neq A$ a finite λ*-open set. Then $A \cap (\bigcap_{\alpha\in I} B_\alpha)$ is a finite λ*-open set, where λ is λ-regular.

Proof. We see that there exists an integer n such that $A \cap (\bigcap_{\alpha\in I} B_\alpha) = A \cap (\bigcap_{i=1}^{n} B_{\alpha i})$ and hence we have the result.

Using Proposition 4.4, we can prove the following:

Theorem 4.5. Let X be a space and for any $\alpha \in I$, B_α a λ*-open set and for any $\beta \in J$, B_β a nonempty finite λ*-open set. Then $(\bigcup_{\beta\in J} B_\beta) \cap (\bigcap_{\alpha\in I} B_\alpha)$ is a λ* open set, where λ is λ-regular.

5. Applications

Let A be a nonempty finite λ*-open set. It is clear, by Proposition 3.3 and Proposition 4.1, that if λ is λ-regular, then there exists a natural number m such that $\{A_1, A_2, \ldots, A_m\}$ is the class of all minimal λ*-open sets in A satisfying the following two conditions:

(1) For any ι, n with $1 \leq \iota, n \leq m$ and $\iota \neq n$, $A_\iota \cap A_n = \phi$.

(2) If C is a minimal λ*-open set in A, then there exists ι with $1 \subseteq \iota \subseteq m$ such that $C = A_\iota$.

Theorem 5.1. Let X be a space and $\phi \neq A$ a finite λ*-open set such that A is not a minimal λ*-open set. Let $\{A_1, A_2, \ldots, A_m\}$ be a class of all minimal λ*-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup \ldots \cup A_m)$. Define $A_y = \bigcap\{B: B$ is λ*-open neighborhood of $x\}$. Then there exists a natural number $k \in \{1,2,3,\ldots,m\}$ such that A_k is contained in A_y, where λ is λ-regular.

Proof. Suppose on the contrary that for any natural number $k \in \{1,2,3,\ldots,m\}$, A_k is not contained in A_y. By Corollary 3.7, for any minimal λ*-open set A_k in A, $A_k \cap A_y = \phi$. By Proposition 4.4, $\phi \neq A_y$ is a finite λ*-open set. Therefore by Proposition 4.1, there exists a minimal λ*-open set C such that $C \subseteq A_y$. Since $C \subseteq A_y \subseteq A$, we have C is a minimal λ*-open set in A. By supposition, for any minimal λ*-open set A_k, we have $A_k \cap C \subseteq A_k \cap A_y = \phi$. Therefore for any natural number $k \in \{1,2,3,\ldots,m\}$, $C \neq A_k$. This contradicts our assumption. Hence the proof.

Proposition 5.2. Let X be a space and $\phi \neq A$ be a finite λ*-open set which is not a minimal λ*-open set. Let $\{A_1, A_2, \ldots, A_m\}$ be a class of all minimal λ*-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup \ldots \cup A_m)$. Then there exists a natural number $k \in \{1,2,3,\ldots,m\}$, such that for any λ*-open neighborhood B_y of y, A_k is contained in B_y, where λ is λ-regular.

Proof. This follows from Theorem 5.1, as $\bigcap\{B: B$ is λ*-open of $y\} \subseteq B_y$. Hence the proof.

Theorem 5.3. Let X be a space and $\phi \neq A$ be a finite λ*-open set which is not a minimal λ*-open set. Let $\{A_1, A_2, \ldots, A_m\}$ be the class of all minimal λ*-open sets in A and $y \in A \setminus (A_1 \cup A_2 \cup \ldots \cup A_m)$. Then there exists a natural number $k \in \{1,2,3,\ldots,m\}$, such that $y \in \lambda^*Cl(A_k)$. where λ is λ-regular.

Proof. It follows from Proposition 5.2, that there exists a natural number $k \in \{1,2,3,\ldots,m\}$ such that $A_k \subseteq B$ for any λ*-open neighborhood B of y. Therefore $\phi \neq A_k \cap A_k \subseteq A_k \cap B$ implies $y \in \lambda^*Cl(A_k)$. This completes the proof.

Proposition 5.4. Let $\phi \neq A$ be a finite λ*-open set in a space X and for each $k \in \{1,2,3,\ldots,m\}$, A_k is a minimal λ*-open sets in A. If the class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal λ*-open sets in A, then for any $\phi \neq B_k \subseteq A_k$, $A \subseteq \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$, where λ is λ-regular.

Proof. If A is a minimal λ*-open set, then this is the result of Theorem 3.11 (2). Otherwise A is not a minimal λ*-open set. If x is any element of $A \setminus (A_1 \cup A_2 \cup \ldots \cup A_m)$, we have $x \in \lambda^*Cl(A_1) \cup \lambda^*Cl(A_2) \cup \ldots \cup \lambda^*Cl(A_m)$ by Theorem 5.3. Therefore $A \subseteq \lambda^*Cl(A_1) \cup \lambda^*Cl(A_2) \cup \ldots \cup \lambda^*Cl(A_m) = \lambda^*Cl(B_1) \cup \lambda^*Cl(B_2) \cup \ldots \cup \lambda^*Cl(B_m) = \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$ by Theorem 3.11 (3).

Proposition 5.5. Let $\phi \neq A$ be a finite λ*-open set and A_k is a minimal λ*-open set in A, for each $k \in \{1,2,3,\ldots,m\}$. If for any $\phi \neq B_k \subseteq A_k$, $A \subseteq \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$ then $\lambda^*Cl(A) = \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$.

Proof. For any $\phi \neq B_k \subseteq A_k$ with $k \in \{1,2,3,\ldots,m\}$, we have $\lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m) \subseteq \lambda^*Cl(A)$. Also, we have $\lambda^*Cl(A) \subseteq \lambda^*Cl(B_1) \cup \lambda^*Cl(B_2) \cup \ldots \cup \lambda^*Cl(B_m) = \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$. Therefore $\lambda^*Cl(A) =$

$\lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$ for any nonempty subset B_k of A_k with $k \in \{1,2,3,\ldots,m\}$.

Proposition 5.6. Let $\phi \neq A$ be a finite λ^*-open set and for each $k \in \{1,2,3,\ldots,m\}$, A_k is a minimal λ^*-open set in A. If for any $\phi \neq B_k \subseteq A_k$, $\lambda^*Cl(A) = \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$, then the class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal λ^*-open sets in A.

Proof. Suppose that C is a minimal λ^*-open set in A and $C \neq A_k$ for $k \in \{1,2,3,\ldots,m\}$, Then we have $C \cap \lambda^*Cl(A_k) = \phi$ for each $k \in \{1,2,3,\ldots,m\}$. It follows that any element of C is not contained in $\lambda^*Cl(A_1 \cup A_2 \cup \ldots \cup A_m)$. This is a contradiction to the fact that $C \subseteq A \subseteq \lambda^*Cl(A) = \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$. This completes the proof.

Combining Proposition 5.4, 5.5 and 5.6, we have the following theorem:

Theorem 5.7. Let A be a nonempty finite λ^*-open set and A_k a minimal λ^*-open set in A for each $k \in \{1,2,3,\ldots,m\}$. Then the following three conditions are equivalent:

(1) The class $\{A_1, A_2, \ldots, A_m\}$ contains all minimal λ^*-open sets in A.

(2) For any $\phi \neq B_k \subseteq A_k$, $A \subseteq \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$.

(3) For any $\phi \neq B_k \subseteq A_k$, $\lambda^*Cl(A) = \lambda^*Cl(B_1 \cup B_2 \cup B_3 \cup \ldots \cup B_m)$, where λ is λ-regular.

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